

YITP-96-10

May 1996

Free Field Representation of Level- k
 Yangian Double $\mathcal{DY}(sl_2)_k$
 and Deformation of Wakimoto Modules[#]

Hitoshi Konno[†]

*Yukawa Institute for Theoretical Physics,
 Kyoto University, Kyoto 606-01, Japan*^{*}

ABSTRACT

Free field representation of level- $k(\neq 0, -2)$ Yangian double $\mathcal{DY}(sl_2)_k$ and a corresponding deformation of Wakimoto modules are presented. We also realize two types of vertex operators intertwining these modules.

[#] Accepted for publication in Lett.Math.Phys.

[†] Yukawa fellow

^{*} E-mail: konno@yukawa.kyoto-u.ac.jp

1 Introduction

Yangian is a Hopf algebra associated with the rational solutions of the quantum Yang-Baxter equation (YBE) [1]. In [2, 3, 4], it has been shown that the Yangian double $\mathcal{DY}(g)$, a quantum double of the Yangian [5], is a relevant object to describe a mathematical structure of massive integrable quantum field theory. However, this idea has not yet been realized successfully. It seems that this is because only level-0 Yangian double has been considered, where infinite dimensional representations are lacking. This situation makes a contrast to the success in integrable spin chains, where the quantum affine algebras, a Hopf algebra associated with the trigonometric solutions of the quantum YBE, have been applied [6, 7].

Recently, a breakthrough has been brought by Iohara, Kohno [8, 9] and Khoroshkin, Lebedev, Pakuliak [10, 11]. They have succeeded to define a central extension of the Yangian double $\mathcal{DY}(sl_n)$ by means of the Drinfeld currents [12] and to obtain infinite dimensional representations which are analogous to the level-1 highest weight representations of the affine Lie algebra \widehat{sl}_n and the quantum affine algebra $U_q(\widehat{sl}_n)$.

In this letter, we extend this line to the higher level representation for the central extension of $\mathcal{DY}(sl_2)$. We consider a representation of the level- k ($\neq 0, -2$) Drinfeld currents in terms of the three free bosonic fields Φ, ϕ and χ . This can be regarded as a deformation of the so-called Wakimoto construction of the affine Lie algebras [13]. As a consequence, an analogue of the level- k Wakimoto modules are obtained as a certain restriction of the Fock spaces of these bosons. We also give a boson representation of vertex operators intertwining the resultant modules.

This paper is organized as follows. In section 2, we briefly recall the definition of the central extension of the Yangian double $\mathcal{DY}(sl_2)$. In section 3, introducing three bosonic fields we give a free field construction of the level- k Drinfeld currents. In section 4, we point out the existence of a couple of operators called screening charges which commutes with all the currents. Their free field representation is also given. In section 5, we consider vertex operators called type I and type II in the terminology by Jimbo and Miwa [7] and give their bosonized expressions. The final section is devoted to discussions.

2 Central extension of the Yangian double

Definition 2.1 The central extension of the yangian double $\mathcal{DY}(sl_2)$ [8, 10] is a Hopf algebra over $\mathbf{C}[[\hbar]]$ generated by the derivation operator d , central element c and the symbols e_m, f_m, h_m , $m \in \mathbf{Z}$ satisfying the following relations.

$$\begin{aligned}
[d, e(u)] &= \frac{d}{du}e(u), & [d, f(u)] &= \frac{d}{du}f(u), & [d, h^\pm(u)] &= \frac{d}{du}h^\pm(u), \\
e(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar}e(v)e(u), \\
f(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar}f(v)f(u), \\
h^\pm(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar}e(v)h^\pm(u), \\
h^+(u)f(v) &= \frac{u-v-(1+c)\hbar}{u-v+(1-c)\hbar}f(v)h^+(u), \\
h^-(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar}f(v)h^-(u), \\
[h^\pm(u), h^\pm(v)] &= 0, \\
h^+(u)h^-(v) &= \frac{u-v+\hbar}{u-v-\hbar} \frac{u-v-(1+c)\hbar}{u-v+(1-c)\hbar} h^-(v)h^+(u), \\
[e(u), f(v)] &= \frac{1}{\hbar}(\delta(u-(v+\hbar c))h^+(u) - \delta(u-v)h^-(v)),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
e^\pm(u) &= \pm \sum_{\substack{m \geq 0 \\ m < 0}} e_m u^{-m-1}, & f^\pm(u) &= \pm \sum_{\substack{m \geq 0 \\ m < 0}} f_m u^{-m-1}, \\
h^\pm(u) &= 1 \pm \hbar \sum_{\substack{m \geq 0 \\ m < 0}} h_m u^{-m-1},
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
e(u) &= e^+(u) - e^-(u), & f(u) &= f^+(u) - f^-(u), \\
\delta(u-v) &= \sum_{n+m=-1} u^n v^m.
\end{aligned}$$

The coproduct is given by[14]

$$\Delta(d) = d \otimes 1 + 1 \otimes d, \quad \Delta(c) = c \otimes 1 + 1 \otimes c,$$

$$\begin{aligned}
\Delta(e^\epsilon(u)) &= e^\epsilon(u) \otimes 1 \\
&+ \left[1 \otimes 1 + \hbar^2 f^\epsilon(u + \hbar - \delta_{\epsilon,+} c \hbar) \otimes e^\epsilon(u - \frac{1}{2} \epsilon c_1 \hbar) \right]^{-1} h^\epsilon(u) \otimes e^\epsilon(u - \frac{1}{2} \epsilon c_1 \hbar), \\
\Delta(h^\epsilon(u)) &= \left[1 \otimes 1 + \hbar^2 f^\epsilon(u + \hbar - \delta_{\epsilon,+} c \hbar) \otimes e^\epsilon(u - \hbar - \frac{1}{2} \epsilon c_1 \hbar) \right]^{-2} \\
&\times h^\epsilon(u) \otimes h^\epsilon(u - \frac{1}{2} \epsilon c_1 \hbar), \\
\Delta(f^\epsilon(u)) &= 1 \otimes f^\epsilon(u + \frac{1}{2} c_1 \hbar) \\
&+ \left[1 \otimes 1 + \hbar^2 f^\epsilon(u + \delta_{\epsilon,+} c_2 \hbar) \otimes e^\epsilon(u - \hbar + \frac{1}{2} c_1 \hbar + \delta_{\epsilon,+} c \hbar) \right]^{-1} \\
&\times f^\epsilon(u + \delta_{\epsilon,+} c_2 \hbar) \otimes h^\epsilon(u + \frac{1}{2} c_1 \hbar + \delta_{\epsilon,+} c \hbar),
\end{aligned} \tag{2.3}$$

where $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$, $\epsilon = \pm$, $\delta_{+,+} = 1$ and $\delta_{+,-} = 0$.

Remark. The formulae (2.1) and (2.3) are converted to those in [8] by the following rule[14].

$$H^\pm(u \mp \frac{1}{4} c \hbar) = h^\pm(u), \quad E(u) = e(u), \quad F(u + \frac{1}{2} c \hbar) = f(u). \tag{2.4}$$

Note also

$$\begin{aligned}
h^\epsilon(u) e^\epsilon(u + \hbar) &= e^\epsilon(u - \hbar) h^\epsilon(u), \\
h^\epsilon(u) f^\epsilon(u - \hbar - \delta_{\epsilon,+} c \hbar) &= f^\epsilon(u + \hbar - \delta_{\epsilon,+} c \hbar) h^\epsilon(u).
\end{aligned} \tag{2.5}$$

3 Free bosons

Let us introduce a Heisenberg algebra generated by $a_{\Phi,n}$, $a_{\phi,n}$, $a_{\chi,n}$ $n \in \mathbf{Z}$ and $a_\Phi, a_\phi, a_\chi, \partial_\Phi, \partial_\phi, \partial_\chi$ satisfying the commutation relations

$$\begin{aligned}
[a_{\Phi,m}, a_{\Phi,n}] &= \frac{k+2}{2} m \delta_{m+n,0}, & [\partial_\Phi, a_\Phi] &= \frac{k+2}{2}, \\
[a_{\phi,m}, a_{\phi,n}] &= -m \delta_{m+n,0}, & [\partial_\phi, a_\phi] &= -1, \\
[a_{\chi,m}, a_{\chi,n}] &= m \delta_{m+n,0}, & [\partial_\chi, a_\chi] &= 1 \\
[a_{X,m}, a_{X',n}] &= 0, & X &\neq X'
\end{aligned} \tag{3.1}$$

with $k \in \mathbf{C}(\neq 0, -2)$.

Let $|l; s, t\rangle$ be a vacuum state with Φ, ϕ, χ -charges $l, -s, t$:

$$\begin{aligned} |l; s, t\rangle &= e^{\frac{l}{k+2}a_\Phi + sa_\phi + ta_\chi} |0\rangle, \\ a_{X,n}|0\rangle &= 0, \quad n > 0, \quad \partial_X|0\rangle = 0 \end{aligned} \quad (3.2)$$

for $X = \Phi, \phi, \chi$. Let $\mathcal{F}_{l,s,t}$ be a Fock space constructed on $|l; s, t\rangle$:

$$\mathcal{F}_{l,s,t} = \left\{ \prod a_{\Phi,-n} \prod a_{\phi,-n'} \prod a_{\chi,-n''} |l; s, t\rangle \right\} \quad (3.3)$$

with $n, n', n'' \in \mathbf{Z}_{>0}$.

It is convenient to introduce generating functions $X(u; A, B)$, $X = \Phi, \phi, \chi$

$$\begin{aligned} X(u; A, B) &= \sum_{n>0} \frac{a_{X,-n}}{n} (u + A\hbar)^n - \sum_{n>0} \frac{a_{X,n}}{n} (u + B\hbar)^{-n} \\ &\quad + \log(u + B\hbar) \partial_X + a_X. \end{aligned} \quad (3.4)$$

We also use the notation $X(u; A, A) = X(u; A)$. We denote by $:$ the normal ordered product

$$\begin{aligned} : \exp\{X(u; A, B)\} &:= \exp\left\{ \sum_{n>0} \frac{a_{X,-n}}{n} (u + A\hbar)^n \right\} e^{a_X} \\ &\quad \times (u + B\hbar)^{\partial_X} \exp\left\{ - \sum_{n>0} \frac{a_{X,n}}{n} (u + B\hbar)^{-n} \right\} \end{aligned}$$

and by ${}_\alpha \partial_u$ the difference operator

$${}_\alpha \partial_u g(u) = \frac{g(u + \alpha\hbar) - g(u)}{\hbar}.$$

Definition 3.1 The currents $e(u), f(u)$ and $h^\pm(u)$ acting on $\mathcal{F}_{l,s,t}$ are defined by

$$e(u) = - : [{}_1 \partial_u \exp\{-\chi(u; -(k+2))\}] \exp\{-\phi(u; -(k+1), -(k+2))\} : \quad (3.5)$$

$$\begin{aligned} f(u) &= \frac{1}{\hbar} : \left[\exp\left\{ \sum_{n>0} \frac{a_{\Phi,n}}{n} [(u - 2\hbar)^{-n} - u^{-n}] \right\} \left(\frac{u}{u - 2\hbar} \right)^{\partial_\Phi} \right. \\ &\quad \times \exp\{\phi(u; -1, 0) + \chi(u; -1)\} \\ &\quad \left. - \exp\left\{ \sum_{n>0} \frac{2a_{\Phi,-n}}{(k+2)n} [(u - (k+3)\hbar)^n - (u - \hbar)^n] \right\} \right] \end{aligned}$$

$$\times \exp\{\phi(u; -(k+3), -(k+2)) + \chi(u; -(k+2))\} : \quad (3.6)$$

$$\begin{aligned} h^+(u) &= \exp\left\{\sum_{n>0} \frac{a_{\Phi,n}}{n} [(u - (k+2)\hbar)^{-n} - (u - k\hbar)^{-n}]\right\} \left(\frac{u - k\hbar}{u - (k+2)\hbar}\right)^{\partial_{\Phi}} \\ &\times \exp\left\{\sum_{n>0} \frac{a_{\phi,n}}{n} [(u - (k+2)\hbar)^{-n} - (u - k\hbar)^{-n}]\right\} \left(\frac{u - k\hbar}{u - (k+2)\hbar}\right)^{\partial_{\phi}} \end{aligned} \quad (3.7)$$

$$\begin{aligned} h^-(u) &= \exp\left\{\sum_{n>0} \frac{2a_{\Phi,-n}}{(k+2)n} [(u - (k+3)\hbar)^n - (u - \hbar)^n]\right\} \\ &\times \exp\left\{\sum_{n>0} \frac{a_{\phi,-n}}{n} [(u - (k+3)\hbar)^n - (u - (k+1)\hbar)^n]\right\} \end{aligned} \quad (3.8)$$

We also define the operator d by

$$\begin{aligned} d &= d_{\Phi} + d_{\phi} + d_{\chi}, \quad (3.9) \\ d_{\Phi} &= \frac{2}{k+2} \left(a_{\Phi,-1} \partial_{\Phi} + \sum_{n \in \mathbf{Z}_{>0}} a_{\Phi,-(n+1)} a_{\Phi,n} \right), \\ d_{\phi} &= -a_{\phi,-1} \partial_{\phi} - \sum_{n \in \mathbf{Z}_{>0}} a_{\phi,-(n+1)} a_{\phi,n}, \\ d_{\chi} &= a_{\chi,-1} \partial_{\chi} + \sum_{n \in \mathbf{Z}_{>0}} a_{\chi,-(n+1)} a_{\chi,n}. \end{aligned}$$

Proposition 3.1 *For $X = \Phi, \phi, \chi$,*

$$\begin{aligned} e^{\gamma d} a_{X,-m} e^{-\gamma d} &= \sum_{n \geq 0} \frac{(m+n-1)!}{(m-1)!n!} \gamma^n a_{X,-(m+n)}, \quad n \geq 1 \\ e^{\gamma d} a_{X,m} e^{-\gamma d} &= \sum_{0 \leq n < m} \frac{(-)^n m!}{(m-n)!n!} \gamma^n a_{X,m-n} + (-)^m \gamma^m \partial_X, \quad n \geq 1 \\ e^{\gamma d} a_X e^{-\gamma d} &= a_X + \sum_{n \geq 1} \frac{a_{X,-n}}{n} \gamma^n, \quad e^{\gamma d} \partial_X e^{-\gamma d} = \partial_X \end{aligned}$$

By direct calculation using (3.5)~(3.9), we have

Theorem 3.1 *By analytic continuation, the operator d and the currents $e(u), f(u)$ satisfy the same relations as (2.1) with $c = k$ on $\mathcal{F}_{l,s,t}$.*

We denote by $\mathcal{DY}(sl_2)_k$ the algebra generated by the currents $e(u), f(u), h^{\pm}(u)$ and d on $\mathcal{F}_{l,s,t}$.

Remark. In the classical limit $\hbar \rightarrow 0$, the currents $e(u), f(u), h^\pm(u)$ tend to the bosonized form of the level- k currents for the affine Lie algebra \widehat{sl}_2 :

$$\begin{aligned} e(u) &\rightarrow -\left[\partial e^{-\chi(u)}\right]e^{-\phi(u)}, \\ f(u) &\rightarrow \left[(k+2)\partial\phi(u) + (k+1)\partial\chi(u) + 2\partial\Phi(u)\right]e^{\chi(u)+\phi(u)}, \\ \frac{1}{\hbar}\left(h^+(u) - h^-(u)\right) &\rightarrow 2\left(\partial\phi(u) + \partial\Phi(u)\right) \end{aligned}$$

with

$$X(u) = a_X + (\log u)\partial_X - \sum_{n \in \mathbf{Z}(\neq 0)} \frac{a_{X,n}}{n} u^{-n}, \quad X = \Phi, \phi, \chi.$$

In the same limit, the operator d is nothing but the L_{-1} operator of the Virasoro algebra constructed via Sugawara form from the above classical currents.

For the level- k quantum affine algebra $U_q(\widehat{sl}_2)$, an analogous free field representation was obtained in [15, 16, 17, 18].

4 Deformed Wakimoto modules

It is easy to show

Proposition 4.1

$$[e(u), \partial_\phi + \partial_\chi] = 0, \quad [h^\pm(u), \partial_\phi + \partial_\chi] = 0, \quad [f(u), \partial_\phi + \partial_\chi] = 0.$$

We hence restrict the Fock space $\mathcal{F}_{l,s,t}$ to the $s = t$ sector without loss of generality.

Remark. On $\mathcal{F}_{l,s,s}$, the currents $e(u), f(u)$ and $h^\pm(u)$ are single valued so that the expansion such as (2.2) makes sense.

Let us next consider the following operators.

$$\xi(u) =: \exp\{-\chi(u; -(k+2))\} :, \quad \eta(u) =: \exp\{\chi(u; -(k+2))\} :$$

We have

$$\xi(u)\eta(v) = -\eta(v)\xi(u) \sim \frac{1}{u-v}. \quad (4.1)$$

Here \sim means that the relation holds modulo regular terms. The fields $\xi(u)$ and $\eta(u)$ are single valued on $\mathcal{F}_{l,s,s}$ $s \in \mathbf{Z}$. From (4.1), the zero-modes $\xi_0 = \oint \frac{du}{2\pi i u} \xi(u)$ and $\eta_0 = \oint \frac{du}{2\pi i} \eta(u)$ anticommute $\{\xi_0, \eta_0\} = 0$. Note also $\xi_0^2 = 0 = \eta_0^2$. In addition, the following equations hold in the sense of analytic continuation.

Proposition 4.2

$$\begin{aligned} h^\pm(u)\eta(v) &= \eta(v)h^\pm(u) \sim 0, \\ f(u)\eta(v) &= -\eta(v)f(u) \sim 0, \\ e(u)\eta(v) &= -\eta(v)e(u) \\ &\sim {}_1\partial_v \left(\frac{1}{u-v+\hbar} \exp\{-\phi(u; -(k+2), -(k+3))\} \right). \end{aligned}$$

Therefore the zero-mode η_0 commutes with the action of $\mathcal{DY}(sl_2)_k$.

From this we restrict the Fock space $\mathcal{F}_{l,s,s}$ to the kernel of η_0 . We hence arrive at the deformation of the Wakimoto modules

$$\mathcal{F}_l = \bigoplus_{s \in \mathbf{Z}} \text{Ker}(\eta_0 : \mathcal{F}_{l,s,s} \rightarrow \mathcal{F}_{l,s,s+1}). \quad (4.2)$$

For the level $-k$ $U_q(\widehat{sl_2})$, the q -deformation of the Wakimoto modules were obtained in [19, 20, 21].

Remark. For $m \geq 0$

$$\begin{aligned} e_m|l; 0, 0\rangle &= \oint \frac{du}{2\pi i} u^m e(u)|l; 0, 0\rangle \\ &= \oint \frac{du}{2\pi i} u^m \frac{1}{\hbar} \left(e^{-\sum_{n>0} \frac{a\chi_{-n}}{n} (u-(k+1)\hbar)^n} - e^{-\sum_{n>0} \frac{a\chi_{-n}}{n} (u-(k+2)\hbar)^n} \right) \\ &\quad \times e^{-\sum_{n>0} \frac{a\phi_{-n}}{n} (u-(k+1)\hbar)^n} |l; 0, 0\rangle \\ &= 0. \end{aligned} \quad (4.3)$$

In the same way, for $m \geq 0$,

$$\begin{aligned} f_m|l; 0, 0\rangle &= \delta_{m,0} 2l|l; 0, 0\rangle \\ &+ \sum_{a=2}^{\infty} \binom{l+a-1}{a} 2^a \hbar^{a-1} \\ &\times \oint \frac{du}{2\pi i} u^{m-a} e^{\sum_{n>0} \frac{a\chi_{-n}}{n} (u-\hbar)^n + \sum_{n>0} \frac{a\phi_{-n}}{n} (u-\hbar)^n} |l; 0, 0\rangle, \quad l \neq 0, \end{aligned}$$

$$\begin{aligned}
f_m|0;0,0\rangle &= 0, \\
h_m|l;0,0\rangle &= \delta_{m,0} \frac{2l}{k+2} |l;0,0\rangle \\
&+ (1 - \delta_{m,0}) ((k+2)\hbar)^m \sum_{\substack{a=0 \\ m+1 \geq a}}^l \frac{l(l+m-a)!}{a!(l-a)!(m-a+1)!} \left(\frac{-k}{k+2}\right)^a |l;0,0\rangle.
\end{aligned}$$

Hence the states $|l;0,0\rangle$ are not the highest weight states in the usual sense except for the case $l = 0$. However, $|l;0,0\rangle$ $l \neq 0$ recovers the highest weight state conditions in the limit $\hbar \rightarrow 0$. We hence identify the state $|l;0,0\rangle$ with the highest weight state $|\lambda_l\rangle$ in the affine Lie algebra \widehat{sl}_2 , where $\lambda_l = (k-l)\Lambda_0 + l\Lambda_1$ are dominant integral weights of level $-k$ with Λ_0 and Λ_1 being the fundamental weights. This identification is natural, because the zero-modes of the free bosons (3.1) are not affected by the deformations.

Next we consider the screening operator defined by

$$\begin{aligned}
S(u)_{[J]} &=: [_1\partial_u \exp\{-\chi(u; -1)\}] \exp\{-\phi(u; -1, 0)\} : \\
&\times \exp\left\{-\sum_{n>0} \frac{2a_{\Phi,-n}}{(k+2)n} (u-\hbar)^n\right\} e^{-\frac{2}{k+2}a_{\Phi}} \prod_{j=0}^J \left(\frac{u-(2+(k+2)j)\hbar}{u-(k+2)j\hbar}\right)^{\partial_{\Phi}} \\
&\times \exp\left\{\sum_{j=0}^J \sum_{n>0} \frac{a_{\Phi,n}}{n} [(u-(k+2)j\hbar)^{-n} - (u-(2+(k+2)j)\hbar)^{-n}]\right\}
\end{aligned} \tag{4.4}$$

with $J \in \mathbf{Z}_{>0}$.

We have

Proposition 4.3

$$\begin{aligned}
h^+(u)S(v)_{[J]} &= S(v)_{[J]}h^+(u) \sim 0, \\
e(u)S(v)_{[J]} &= S(v)_{[J]}e(u) \sim 0, \\
S(u)_{[J]}\eta(v) &= \eta(v)S(u)_{[J]} \\
&\sim -_1\partial_v \left(\frac{1}{u-v+(k+2)\hbar} \exp\{-\phi(v; -(k+3), -(k+2))\}\right) \\
&\times \exp\left\{-\sum_{n>0} \frac{2a_{\Phi,-n}}{(k+2)n} v^n\right\} e^{-\frac{2}{k+2}a_{\Phi}} \prod_{j=0}^J \left(\frac{v-(1+(k+2)j)\hbar}{v-(-1+(k+2)j)\hbar}\right)^{\partial_{\Phi}} \\
&\times \exp\left\{\sum_{j=0}^J \sum_{n>0} \frac{a_{\Phi,n}}{n} [(v-(-1+(k+2)j)\hbar)^{-n} - (v-(1+(k+2)j)\hbar)^{-n}]\right\},
\end{aligned}$$

$$[S(u)_{[J]}, \partial_\phi + \partial_\chi] = 0.$$

In addition, in the limit $J \rightarrow \infty$,

$$\begin{aligned} h^-(u)S(v) &= S(v)h^-(u) \sim 0, \\ f(u)S(v) &= S(v)f(u) \\ &\sim -_{k+2}\partial_v \left(\frac{1}{u-v} \exp \left\{ - \sum_{n>0} \frac{2a_{\Phi,-n}}{(k+2)n} (v-\hbar)^n \right\} \right. \\ &\quad \times e^{-\frac{2}{k+2}a_\Phi} \prod_{j=1}^{\infty} \left(\frac{v - (2+(k+2)j)\hbar}{v - (k+2)j\hbar} \right)^{\partial_\Phi} \\ &\quad \left. \times \exp \left\{ \sum_{j=1}^{\infty} \sum_{n>0} \frac{a_{\Phi,n}}{n} [(v - (k+2)j\hbar)^{-n} - (v - (2+(k+2)j)\hbar)^{-n}] \right\} \right), \end{aligned}$$

where $S(u) = \lim_{J \rightarrow \infty} S(u)_{[J]}$.

Therefore the screening charge $S = \oint \frac{du}{2\pi i} S(u)_{[J]}$ commutes with all the currents in $\mathcal{DY}(sl_2)_k$ and η_0 in the limit $J \rightarrow \infty$. The charge S yields a linear map $S : \mathcal{F}_l \rightarrow \mathcal{F}_{l-2}$.

5 Vertex operators

Let $V^{(l)} = \oplus_{m=0}^l \mathbf{C}[[\hbar]]w_m$ be the $l+1$ -dim representation of sl_2 . Let $V_u^{(l)} = V^{(l)} \otimes \mathbf{C}[[\hbar]][[u^{-1}, u]]$ be the evaluation module on which the action of $\mathcal{DY}(sl_2)_k$ is defined by the following relations[22].

$$\begin{aligned} e_n(w_m \otimes u^a) &= m \left(u + \frac{l-2m+1}{2}\hbar \right)^n (w_{m-1} \otimes u^a), \\ f_n(w_m \otimes u^a) &= (l-m) \left(u + \frac{l-2m-1}{2}\hbar \right)^n (w_{m+1} \otimes u^a), \\ h_n(w_m \otimes u^a) &= \left[(m+1)(l-m) \left(u + \frac{l-2m-1}{2}\hbar \right)^n \right. \\ &\quad \left. - m(l-m+1) \left(u + \frac{l-2m+1}{2}\hbar \right)^n \right] (w_m \otimes u^a), \\ d(w_m \otimes u^a) &= -a(w_m \otimes u^{a-1}) \end{aligned} \tag{5.1}$$

for $a, n \in \mathbf{Z}$, where we set $w_{-1} = w_{l+1} = 0$.

Let λ_l be the dominant integral weights.

Definition 5.1 The type I and the type II vertex operators are the following intertwiners.

$$\text{Type I} \quad \Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) : \mathcal{F}_{l_1} \rightarrow \mathcal{F}_{l_3} \otimes V_u^{(l)},$$

$$\text{Type II} \quad \Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) : \mathcal{F}_{l_1} \rightarrow V_u^{(l)} \otimes \mathcal{F}_{l_3}.$$

satisfying

$$\begin{aligned} \Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)x &= \Delta(x)\Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u), \\ \Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)x &= \Delta(x)\Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u), \quad \text{for } \forall x \in \mathcal{DY}(sl_2)_k. \end{aligned} \quad (5.2)$$

We normalize the vertex operators as follows.

$$\Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)|\lambda_{l_1}\rangle = |\lambda_{l_3}\rangle \otimes w_m + \cdots, \quad (5.3)$$

$$\Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)|\lambda_{l_1}\rangle = w_m \otimes |\lambda_{l_3}\rangle + \cdots \quad (5.4)$$

with $m = (l + l_3 - l_1)/2$. Expanding $\Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)$ and $\Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u)$ in a formal series

$$\Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) = \sum_{m=0}^l \Phi_{l,m}(u) \otimes w_m, \quad (5.5)$$

$$\Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) = \sum_{m=0}^l w_m \otimes \Psi_{l,m}(u), \quad (5.6)$$

we obtain from (2.3) and (5.1) the following intertwining relations.

Lemma 5.1

$$\begin{aligned} \text{Type I :} \quad & (l - m + 1)\Phi_{l,m-1}(u) = [\Phi_{l,m}(u), f_0], \\ & [\Phi_{l,l}(u), e(v)] = 0, \\ & h^+(v)\Phi_{l,l}(u) = \frac{v - u - \frac{k-l+1}{2}\hbar}{v - u - \frac{k+l+1}{2}\hbar} \Phi_{l,l}(u)h^+(v), \\ & h^-(v)\Phi_{l,l}(u) = \frac{v - u + \frac{k+l-1}{2}\hbar}{v - u + \frac{k-l-1}{2}\hbar} \Phi_{l,l}(u)h^-(v). \\ \text{Type II :} \quad & (m + 1)\Psi_{l,m+1}(u) = [\Psi_{l,m}(u), e_0], \end{aligned}$$

$$[\Psi_{l,0}(u), f(v)] = 0,$$

$$h^\pm(v)\Psi_{l,0}(u) = \frac{v-u-\frac{l-1}{2}\hbar}{v-u+\frac{l+1}{2}\hbar}\Psi_{l,0}(u)h^\pm(v),$$

Lemma 5.2 *The following vertex operators satisfy the intertwining relations in Lemma 5.1 in the limit $J \rightarrow \infty$ and $\delta \rightarrow 0$.*

$$\Phi_{l,m}(u)_{[J,\delta]} = \frac{1}{(l-m)!} [\cdots [\Phi_{l,l}(u)_{[J,\delta]}, \underbrace{f_0 \cdots f_0}_{l-m}], \quad (5.7)$$

$$\Psi_{l,m}(u)_{[J,\delta]} = \frac{1}{m!} [\cdots [\Psi_{l,0}(u)_{[J,\delta]}, \underbrace{e_0 \cdots e_0}_m], \quad (m = 0, 1, \dots, l) \quad (5.8)$$

with

$$\begin{aligned} & \Phi_{l,l}(u)_{[J,\delta]} \\ &= \exp \left\{ \sum_{j=0}^J [\Phi^{(-)}(u; -\frac{k+l-1}{2} + 2j; -\frac{k-l-1}{2} + 2j) + \Phi^{(-)}(0; 2 + \delta + 2j; 2j)] \right\} \\ & \quad \times e^{\frac{l}{k+2}a_\Phi} \\ & \quad \times \exp \left\{ -\sum_{j=1}^J \Phi^{(+)}(u; -\frac{k-l+1}{2} - (k+2)j; -\frac{k+l+1}{2} - (k+2)j) \right\}, \quad (5.9) \\ & \Psi_{l,0}(u)_{[J,\delta]} \\ &= \exp \left\{ \sum_{j=0}^J [\Phi^{(-)}(u; \frac{l-5}{2} - k + 2j; -\frac{l+1}{2} - k + 2j) + \Phi^{(-)}(0; 2 + \delta + 2j; 2j)] \right\} \\ & \quad \times e^{\frac{l}{k+2}a_\Phi} \\ & \quad \times \exp \left\{ -\sum_{j=1}^J \Phi^{(+)}(u; -\frac{l-1}{2} - (k+2)j; \frac{l-3}{2} - (k+2)j) \right\} \\ & \quad \times \exp \left\{ \phi(u; \frac{l-5}{2} - k, -\frac{l+3}{2} - k) + \chi(u; \frac{l-3}{2} - k) \right\}, \quad (5.10) \end{aligned}$$

where

$$\Phi^{(-)}(u; A; B) = \sum_{n>0} \frac{2a_{-n}}{(k+2)n} [(u + A\hbar)^n - (u + B\hbar)^n] \quad (5.11)$$

$$\Phi^{(+)}(u; A; B) = \sum_{n>0} \frac{a_n}{n} [(u + A\hbar)^n - (u + B\hbar)^n] - \left[\log \left(\frac{u + A\hbar}{u + B\hbar} \right) \right] \partial_\Phi \quad (5.12)$$

From this and the remark below the proposition 4.3, we arrive at the following theorem.

Theorem 5.1 *The vertex operators*

$$\Phi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) = \sum_{m=0}^l \Phi_{l,m}^{(r)}(u) \otimes w_m, \quad (5.13)$$

$$\Psi_{\lambda_{l_1}}^{\lambda_{l_3} V^{(l)}}(u) = \sum_{m=0}^l w_m \otimes \Psi_{l,m}^{(r)}(u), \quad (5.14)$$

with

$$\begin{aligned} \Phi_{l,m}^{(r)}(u) &= g_{\lambda_{l_1}}^{(I)\lambda_{l_3} V^{(l)}}(u) \oint_{\mathcal{C}} \frac{dt_1}{2\pi i} \oint \frac{dt_2}{2\pi i} \cdots \oint \frac{dt_r}{2\pi i} \Phi_{l,m}(u)_{[J,\delta]} S(t_1)_{[J]} S(t_2)_{[J]} \cdots S(t_r)_{[J]}, \\ \Psi_{l,m}^{(r)}(u) &= g_{\lambda_{l_1}}^{(II)\lambda_{l_3} V^{(l)}}(u) \oint_{\mathcal{C}} \frac{dt_1}{2\pi i} \oint \frac{dt_2}{2\pi i} \cdots \oint \frac{dt_r}{2\pi i} \Psi_{l,m}(u)_{[J,\delta]} S(t_1)_{[J]} S(t_2)_{[J]} \cdots S(t_r)_{[J]}, \end{aligned}$$

where $2r = l_1 + l - l_3$ and $g_{\lambda_{l_1}}^{(I,II)\lambda_{l_3} V^{(l)}}(u)$ being the normalization function determined by the condition (5.3) and (5.4), gives the intertwiner in Definition 5.1 in the limit $J \rightarrow \infty$ and $\delta \rightarrow 0$. The contour \mathcal{C} is depicted in Fig. 1[23].

6 Discussion

We have constructed a free field representation of the level- k Drinfeld currents for the Yangian double $\mathcal{DY}(sl_2)_k$ and screening operators. As a result, we have obtained a deformation of the level- k Wakimoto modules. We also have realized the type I and type II vertex operators intertwining these modules.

A possible application of the results is a calculation of correlation functions in massive integrable quantum field theory such as higher spin $SU(2)$ invariant Thirring model and in higher spin XXX spin chains. For this purpose, one has to make a precise identification of the space of states with the deformed Wakimoto modules. It is also an important problem to derive the deformation of the Knizhnik-Zamolodchikov equation[3, 4] in the framework of free field representation.

We hope to discuss these problems in future publication.

7 Acknowledgments

The author would like to thank Kenji Iohara for stimulating discussion. He is also grateful to Michio Jimbo, Vladimir Korepin, Tetsuji Miwa and Max Niedermaier for discussions. This work is supported by the COE research institute fellowship and the Yukawa Memorial Foundation.

References

- [1] Drinfeld, V.G. Hopf algebras and quantum Yang-Baxter equation. *Soviet Math. Dokl.***32** (1985), 254-258.
- [2] Leclair, A. and Smirnov, F.A. Infinite quantum group symmetry of fields in massive 2D quantum field theory *Int.J.Mod.Phys.***A7**(1992), 2997-3022.
- [3] Smirnov, F.A. Dynamical symmetries of massive integrable models I, II *Int.J.Mod.Phys.***A7** Suppl. **1B** (1992), 813-838, 839-858.
- [4] Bernard, D. and Leclair, A. The quantum double in integrable quantum field theory *Nucl.Phys.***B399**(1993), 709-748.
- [5] Drinfeld, V.G. Quantum groups, Proc. ICM Berkeley (1986), 798-820.
- [6] Frenkel, I.B. and Reshetikhin, N.Yu. Quantum affine algebras and holonomic difference equations. *Commun. Math. Phys.***146** (1992), 1-60.
- [7] Jimbo, M. and Miwa, T. Algebraic Analysis of Solvable Lattice Models. *Conference Board of the Math. Sci., Regional Conference Series in Mathematics*,**85** (1995) and references therein.
- [8] Iohara, K. and Kohno, M. A central extension of $\mathcal{DY}_\hbar(gl_2)$ and its vertex representations. UTMS 95-32, q-alg/9603032. to appear in *Lett. Math. Phys.*.
- [9] Iohara, K. Bosonic Representations of Yangian Double $\mathcal{DY}_\hbar(g)$ with $g = gl_n, sl_n$. Kyoto-Math 96-06, q-alg/9603033, to appear in *Jour. of Phys. A*.
- [10] Khoroshkin, S. Central Extension of the Yangian Double. q-alg/9602031.
- [11] Khoroshkin, S., Lebedev, D., Pakuliak, S. Intertwining Operators for the Central Extension of the Yangian Double. Preprint DFTUZ/95-28, ITEP-TH-15/95, q-alg/9602030.

- [12] Drinfeld, V.G. A new realization of Yangians and quantized affine algebras. *Soviet Math. Dokl.***32** (1988), 212-216.
- [13] Wakimoto, M. Fock space representations of the affine Lie algebra $A_1^{(1)}$ *Comm. Math. Phys.* **104** (1986), 605-609.
- [14] Iohara, K. private communication.
- [15] Shiraishi, J. Free boson representation of $U_q(\widehat{sl_2})$, *Phys. Lett.* **A171**(1992), 243-248.
- [16] Matsuo, A. Free field representation of quantum affine algebra $U_q(\widehat{sl_2})$, *Phys. Lett.* **B308** (1992), 260-.
- [17] Abada, A., Bougourzi, A.H. and Gradechi, El. Deformation of the Wakimoto construction, *Mod. Phys. Lett.* **A8** (1993), 715-724.
- [18] Kato, A., Quano, Y. and Shiraishi, J. Free boson representation of q -vertex operators and their correlation functions, *Comm. Math. Phys.* **157**(1993), 119-137.
- [19] Matsuo, A. A q -deformation of Wakimoto modules, primary fields and screening operators, *Comm. Math. Phys.* **160** (1994), 33-48.
- [20] Konno, H. BRST cohomology in quantum affine algebra $U_q(\widehat{sl_2})$, *Mod. Phys. Lett.***A9**(1994),1253-1265.
- [21] Konno, H. Free field representation of quantum affine algebra $U_q(\widehat{sl_2})$ and form factors in higher spin XXZ model, *Nucl. Phys.***B432**(1994), 457-486.
- [22] Chari, V. and Pressley, A. Yangians and R -matrices, *L'Enseign.Math.t* **36** (1990), 267-302.
- [23] Felder, G. BRST approach to minimal model, *Nucl. Phys.***B317**(1989), 215-236.

This figure "fig1-1.png" is available in "png" format from:

<http://arXiv.org/ps/q-alg/9605045v2>